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NUMBERS, VARIABLES AND MR. RUSSELL'S PHILOSOPHY.

PHILOSOPHERS not infrequently take mathematics as a field for the exercise of their fancies. If the starting-point be a really sound philosophy, the result of an excursion into mathematics enures to the benefit of both disciplines. If, however, the start be a metaphysics or a logic that is essentially erroneous, the results attained may be novel or startling, but have no place in the body of our scientific knowledge. To say that the investigations of any particular philosopher comes under this second head seems invidious, yet it is necessary to so stigmatize the false systems if the way is to be kept open for the true. Of all the impediments to the cultivation of mathematical science on a philosophical basis there can be none greater than the putting forth a pseudo-philosophy in the guise of true doctrine. As such an impediment do we esteem the work of Mr. Bertrand Russell, who is a recent writer of some repute, though whether he ought to be classed as philosopher who devotes himself to mathematics or as a mathematician who dips into philosophy is a moot question.

Mr. Russell's labors seem to have as their burden the reorganization of mathematics upon the basis of what he is pleased to term "one of the greatest discoveries of our age"; a discovery, if such it be, largely due to Mr. Russell himself (not to speak of his forerunner, Professor Peano) "that all Mathematics is Symbolic Logic."[1] To encounter such a statement is rather startling to any one who has been accustomed to class mathematics among the deduc-

[1] *Principles of Mathematics,* Cambridge, at the University Press, Vol. I, 1913, p. 5.

tive sciences. Symbolic logic is merely deductive logic treated in a particular way, and deductive logic is not usually understood to take within its scope any special deductive science, but is supposed merely to give an account of the methods of deduction employed in these sciences. The view taken by Mr. Russell is tenable only when mathematics and logic are understood to have scopes different from those usually accepted for them. Indeed the mathematical science developed by the school of Peano and Russell has some radical differences, not merely in scope but also in method, from what is ordinarily expounded under the name of mathematics. These neo-mathematicians hold that mathematics ought not to follow the method of laying down a set of special mathematical axioms and postulates for each branch of the science and deducting therefrom the theorems of that branch. The only principles Mr. Russell would have pure mathematics put forword are "ten principles of deduction and ten other premises of a general logical nature."[2] The natural result is that what Mr. Russell attains are not the theorems of ordinary mathematics. Where ordinary mathematics would deduce from the axioms A, B and C the theorem T, Mr. Russell is satisfied to have mathematics not assert A or B or C at all—much less T—but merely prove and assert the proposition that "A, B and C imply T"; in other words that the theorem T is a logical consequence of A, B and C if these are given as premises.

To replace the ordinary theorems of mathematics by propositions of implication is alone insufficient to bring mathematics into the realm of symbolic logic. Mr. Russell goes still further and so extends the denotations of the mathematical symbols made use of that the propositions of pure mathematics become the merest of shells. A mathematical formula where x and y originally meant quantities

[2] P. 4.

is by Mr. Russell so extended that x and y may have as wide a range of applicability as the X and Y of the canonical logical proposition: "Every X is a Y." Thus if by some far-fetched interpretation of addition, multiplication etc. the formula $(x+y)^2 = x^2 + 2xy + y^2$ can be made to yield a significance when x and y instead of meaning quantities are understood to mean Plato and Socrates respectively, Mr. Russell would bring the formula as thus construed within the mathematical field. Pure mathematics then, with Mr. Russell, is characterized, not by the import of its propositions, but by their form, and its end in each question it takes up is merely to find what form of proposition is implied by a set of propositions of specified forms. It thus becomes identified with symbolic logic provided deductive logic be held to deal, not merely with generalities in the theory of inference, but to have an essential part of its scope the consideration of every possible combination of the types of propositions which it accepts, and to endeavor in each such case to find what form of proposition is to be taken as the type of conclusion inferable from this combination of premises.

We shall not here debate the question of how far this new view of the scope of pure mathematics is worthy of being called a "great discovery." But we must point out that Mr. Russell is not entirely consistent in his adhesion to this view. Under it either pure mathematics must be completely identified with symbolic logic or must be put in the rank of a subdivision of the latter discipline. No other alternative is reconcileable with the statement that "all Mathematics is Symbolic Logic." Mr. Russell however, without seeing the need for first retracting this statement, is led by "respect for tradition" and "desire to adhere to usage" to draw a distinction under which logic "consists of the premises of mathematics" together with certain other propositions not mathe-

matical, while mathematics "consists of all the consequences of the above premises" together with some, but not all, of these premises themselves.[3] Obviously then, mathematics will cover a field that is not included in logic at all, and the mathematics of this field can certainly not be symbolic logic.

Notwithstanding the purely formal rôle that Mr. Russell would assign to pure mathematics, he takes up in his *Principles* various questions that seem to pertain to the matter rather than to the form of mathematical science. After enumerating certain "indefinables of mathematics" he proceeds to the definables, and begins by the discussion of number. The doctrine of number is fundamental in mathematical philosophy; let us see how Mr. Russell handles it.

Numbers, Mr. Russell tells us, are "applicable essentially to classes."[4] This word *class* is a favorite one with Mr. Russell, and he often uses it where another word would be more appropriate. In the present case *group* might well be adopted in preference to class. For ordinarily, when reference is made to a class of objects, what is in mind is something about objects of that class taken individually—not about the objects as constituting a collectivity. Now (except when there is but a single object at hand, a case which gives rise to just as much difficulty when we speak of "class" as when we use the word "group") it is precisely this idea of collectivity, so aptly suggested by *group,* that is in evidence when we speak of number. The number of objects in a group is a number that belongs to the group as a whole, not to any of its objects taken separately.

A number then is something belonging to a group—we replace "applicable to" by "belonging to" without stopping to comment on the impropriety of Mr. Russell's use

[3] P. 9. [4] P. 112.

of the former phraseology—and if we follow Mr. Russell we must regard it as a property of that group. "Numbers," he tells us, "are to be regarded as properties of classes."[5] Presumably Mr. Russell does not use "property" in the sense of the old *proprium,* a name by which certain attributes were distinguished from the essential on the one hand and the accidental on the other; in this sense the statement would not be true, for to have a number is of the very essence of a group. Taking, however, property as a mere synonym of attribute, and substituting group for class, no fault need be found with what Mr. Russell says. A number is truly an attribute of a group of objects. But Mr. Russell too hastily proceeds to inquire "Under what circumstances do two classes have the same number?"[6] when the next question ought in fact to be: Do two groups ever really have the *same* number? In a mere mathematician it is pardonable to be unaware that the question of identity of attributes of distinct objects, or distinct groups of objects, is a debatable one, but surely any writer who aspires to be ranked as a philosopher ought to know that eminent thinkers have been at variance in this matter. Thus, going back only a few years, we find that a brief but very interesting discussion took place between Mill and Spencer as to whether two different objects could be said to have the same attribute.

The point really at issue is this: when one speaks of sameness (or identity) in such a case, is he so using language as to mark all the distinctions that ought to be made, or is he ignoring some of them, is he promoting clearness of thought and speech, or is his phraseology pregnant with obscurity and confusion? This question never seems to have occurred to Mr. Russell, though it arises whether the attributes are qualities belonging to different individuals or are number attributes (quantities) belonging to

[5] P. 113. [6] *Ibid.*

different groups. The primary use of *same* or *identical* in connection with attributes is in the case where an object is viewed continuously for a time by an observer who perceives a certain attribute of that object to undergo no change—to be the *same* at the end as at the beginning of the observation. It is in this primary sense of "same" that we speak of the color of an object *remaining the same.*[7] Quite a different case however is at hand when an observer compares two distinct bodies which are before him and decides that they are exactly alike in color, or compares two distinct groups of objects and decides that they are exactly alike (equal) as to number. Colloquially, it is true, "same" would be used here as well as in the previous case; the two bodies would be said to have the same color, and the two groups to contain the same number of objects, but is this colloquial use of "same" worthy of a philosopher? Is it at all suited to the requirements of an exact science? "Same" in its primary sense is used to express one set of facts concerning attributes; why should it be also used to express facts of entirely different character when there are at hand other words, *like* and *equal,* perfectly adapted to convey this second sense? To use "same" and "identical" in this sense is as absurd and misleading as it would be to call two houses exactly alike the same house.

It would be a great mistake to regard the distinction between identity and equality of numbers as a mere verbal subtlety. True, it is not of any moment so far as computation is concerned; it does not affect the compilation of

[7] Under this first head likewise comes the case in which sameness of attribute is asserted where the asserter has not given continuous attention to the object, having viewed it only at the beginning and at the end of an interval of time, but intends to assert that *if* the observation had been continuous no change would have been noted by the observer. A valid claim to classification under another head may however be granted to the case in which change takes place but the ultimate result is precisely the original state of affairs; where, for instance, a body changes color but finally takes on a color attribute exactly like the one it originally possessed—otherwise put, "returns to the same color."

the logarithmic tables used by the engineer or have any bearing on the numerical calculations by which an astronomer predicts an eclipse. But in the philosophy of mathematics the results of this distinction are far reaching. To begin with, it shows us that 1, 2, 3, etc. are general names like "man" and "animal," and not individual names like "Socrates" and "Plato." Mr. Russell's assertion: "It is plain that we cannot take the number 1 itself twice over, for there is one number one and there are not two instances of it,"[8] is quite untenable. Each of the names 1, 2, 3, etc. is a class name belonging to a class containing many numbers, and we may legitimately speak of a one, a two, a three, etc. instead of adhering to the customary locution which omits the article. In none of these classes, which may most aptly be called *value classes,* is any member identical with another member; thus no three is the same as any other three, though both have the same class name, just as both Socrates and Plato belong to the same class, *man,* and alike have "man" as class name.[9]

Since numerals are general names, it follows that such an equation as $5 = 3 + 2$ is not a singular proposition analogous to "Plato was a contemporary of Socrates," but is a universal proposition analogous to "Every man is an animal." It is an assertion about every five, but should not be read: "Every five is equal to every sum of a three plus a two," for each sum of a three and a two is a five, and though equal to every other five cannot be equal to itself. We must read the equation: "Every five is equal to each sum of a three plus a two unless this sum is itself the five in question," or, better, recognize plainly that the so-called

[8] P. 118.

[9] A vague appreciation of the distinction which ought to be drawn between identity and the belonging to the same class appears in the mathematical use of the word *value.* Mathematicians sometimes speak of two numbers as having the same value; a useful and convenient phraseology if "having the same value" means, not identical, but belonging to the same class, while it is a quite inexcusable circumlocution if what be meant is merely that the "two" numbers are identical—are not two numbers but one and the same number.

"sign of equality" does not concern equality alone but has identity as an alternative possibility, and read our equation as "Every five is equal to or identical with each sum of a three plus a two." That =, as used in mathematics, has commonly two alternatives in view seems to have escaped the eyes of mathematicians. Usually it is read as "equals"; Cayley defines an equation to be "an expression or statement of the equality of two quantities."[10] Schroeder however says that "If a and b are any two names for the same number, we write $a = b$. A proposition of this kind is called an equation."[11] According to this, an equation would always have identity in view, and = would never concern equality at all! Weber is likewise wrong in his definition, telling us that an equation is "a proposition which expresses that a symbol a has the same significance as another symbol b, which we express in mathematical symbolism $a = b$."[12] For, to take the above case of $5 = 3 + 2$, it is not true that 5 has the same significance as $3 + 2$ any more than man has the same significance as mathematician. Every sum of a three plus a two is a five, but not every five has been brought into existence by the addition of a two to a three. Recognition of the true doctrine of number will be found to throw a much-needed light, not only upon the interpretation of equations, but also upon many other matters that mathematics has hitherto left in obscurity. We cannot however stop to dwell on the various ramifications of this doctrine, but must pass on to a further consideration of Mr. Russell's philosophy.

Mr. Russell decides that two groups ("classes") have "the same number" when their members can be correlated in a one-to-one relation, and so defines a one-to-one relation as to enable him to include even the case of groups which comprise nothing at all and give rise to the number zero.

[10] *Collected Mathematical Papers*, Vol. II, Art. "Equation."

[11] *Lehrbuch der Arithmetik und Algebra*, Vol. I, p. 23.

[12] *Encyklopädie der elementaren Algebra und Analyse*, p. 17.

He is not satisfied however to rest here, or with the dictum of "Peano and common sense"[13] that when there subsists such a relation (a relation specified as reflexive, symmetrical and transitive) the two groups have a common property called their number. This definition of numbers "by abstraction" is inadequate, he holds, because nothing is laid down by it which would logically bar there being common to the groups many different properties each answering the description, and thus a number is not by the definition uniquely distinguished among the various properties that groups may have in common. Mr. Russell prefers to give what he calls a nominal definition. Premising first that two groups are to be termed *similar* if they can have their objects put into a one-to-one correspondence, and that a group is to be regarded as similar to itself, he defines the number of a group ("class") as "the class of all classes similar to the given class."[14] This is assuredly the most remarkable definition of a number that has ever been penned. It is as though one would define whiteness as the class of all white objects. Mr. Russell himself, though he characterizes it as "an irreproachable definition of the number of a class in purely logical terms,"[14] admits that his definition appears at first sight to be a wholly indefensible paradox, and his attempt to defend it puts it in no more favorable light. He informs us that "when we remember that a class-concept is not itself a collection, but a property by which a collection is defined, we see that if we define the number as the class-concept, not the class, a number is really defined as a common property of a set of similar classes and of nothing else,"[14] but, even were it true that a number was such a property, this contention would be of no avail to palliate the faults of his definition in which the number *is* defined as the class.[15]

[13] P. 114. [14] P. 115.

[15] On page 131 he tells us that "a class cannot be identified with its class-

What does Mr. Russell really mean? On careful consideration we are impelled to conclude that he is merely indulging in what seems to be his favorite vice—confusion between matters that are in fact quite distinct. As justification of his definition, he tells us that such a word as *couple* "obviously does denote a class of classes." The true facts are that "couple" is a class name, and the class to which it belongs is composed of groups. A couple however is not the class of these groups—it is not the totality of couples. If Mr. Russell wished to define the name *totality of couples* or *all couples* his definition of a number would not be inappropriate. But the totality of couples is not a number.[16] A couple taken alone may be termed a number: a concrete number whose number attribute is an abstract number, a two. Mr. Russell makes however no distinction between concrete numbers and abstract numbers; between a group of objects and the number attribute which belongs to that group (and which for distinctiveness had best be called a *natural number*). He seems on the one hand to think that the definition by abstraction, which defines a number as a property common to certain groups, has reference to precisely the same sort of numbers as a definition framed for such numbers as a couple or a trio, while on the other hand he apparently labors under the delusion that a couple is the same thing as the totality of all couples, a trio the same thing as the totality of all trios, etc. He would not, we presume, contend that a soldier is the same thing as a regiment, but he takes a stand which is just as untenable.

Before going any further into questions that belong to mathematics, properly speaking, we must turn back to

concept." We need hardly say that in describing a class-concept as a *property* Mr. Russell is regarding concepts in a light hitherto unknown to philosophy.

[16] At least it is not such a number (a couple) as Mr. Russell has in view. Of course, if the totality of couples be taken as a group whose members are couples, this group, which will comprise millions of members, has just as much right to be regarded as a concrete number as a group composed of two individual members.

consider Mr. Russell's treatment of certain matters that are purely philosophical. Let us begin with what he calls *terms*. "Whatever may be an object of thought," says Mr. Russell, "or may occur in any true or false proposition, or can be counted as *one*, I call a *term*. This then is the widest word in the philosophical vocabulary. I shall use as synonymous with it the words unit, individual, and entity. The first two emphasize the fact that every term is *one,* while the third is derived from the fact that every term has being, i. e., *is* in some sense. A man, a moment, a class, a relation, a chimera, or any thing else that can be mentioned, is sure to be a term; and to deny that such and such a thing is a term must always be false."[17] This "widest word in the philosophical vocabulary" is not however really the widest, it seems, for only twelve pages later Mr. Russell tells us in an inobtrusive note: "I shall use the word *object* in a wider sense than *term,* to cover both singular and plural, and also cases of ambiguity, such as 'a man.' The fact that a word can be framed with a wider meaning than *term* raises grave logical problems." This retraction of what was first said, coming so soon, does not tend to make us believe that Mr. Russell's original remarks concerning "terms" were based upon long and profound thought. And in fact an examination shows Mr. Russell's view of the matter to be very far from what may be demanded of a philosopher.

To survey the question properly we had best begin with names, or rather with substantive words and phrases, and these for purposes of philosophy we may primarily divide into three classes. First come such as are absolutely meaningless, the word *blictri* being the classical example of nouns of this type. Second come substantives so defined as to connote contradictory attributes; as examples we may cite the phrases "honest thief" and "chaste prostitute."

[17] P. 43.

Such a substantive (unless taken in a non-natural sense) cannot be the name of an object of thought, but it ought not to be regarded as entirely meaningless and might conveniently be spoken of as the name of a chimera. Third come names of objects, substantives having meanings attached to them which do not involve the connoting of contradictory attributes. A conception of an object is, of course, itself an object, and likewise the name of an object is an object which by an exceptional use of language (a *suppositio materialis*) may denote itself instead of the object or objects it ordinarily denotes.[18] There is no such object as a chimera (as we propose to use that word), for to say a substantive is the name of a chimera is to assert that there can be no corresponding object of thought. But the name of a chimera, and even a meaningless substantive, and in general every word or phrase, is an object. One of the first steps in philosophy is to distinguish carefully between a name, a conception corresponding to a name, and an object denoted by a name; the last being in exceptional cases the name itself or another name. In framing a philosophical nomenclature one should do all in his power to enforce these distinctions, but it is not too

[18] The distinction between the primary use of a word or phrase and its use in a *suppositio materialis* is quite ancient, dating back to the medieval logicians. Mr. Russell however seems to be serenely unaware of any such distinction, and by ignoring it he is enabled to put forth an argument that must be characterized as only worthy of a schoolboy. He says (p. 48): "It is plain, to begin with, that the concept which occurs in the verbal noun is the very same as that which occurs as verb. This results from the previous argument, that every constituent of every proposition must, on penalty of self-contradiction, be capable of being made a logical subject. If we say '*kills* does not mean the same as *to kill*,' we have already made *kills* a subject, and we cannot say that the concept expressed by the word *kills* cannot be made a subject." Here Mr. Russell, in attempting to show that a verb such as "kills" and a verbal noun such as "to kill" have precisely the same meaning, seeks support in the absurd contention that both words are alike subjicible. The facts are, of course, that "kills" or any other word or phrase can be made a subject by a *suppositio materialis,* but when we consider the *primary* uses of words and phrases there is a clear demarcation between those which can and those which cannot be made logical subjects. In this use, which is the only one relevant to Mr. Russells' argument, "kills" is not, while "to kill" is, subjicible. And the sentence "*kills* does not mean the same as *to kill,*" though making a statement about the primary uses of "kills" and "to kill," is not itself a proposition in which these are given their primary uses, but one in which "kills" and "to kill" each occur in a *suppositio materialis.*

much to say that Mr. Russell, in his nomenclature, gives the impression of being at great pains to obscure them. His use of "term" is a case in point. A philosopher need pay little heed to the colloquial uses of this word, and he may likewise disregard certain authorized applications of it that are not very common. But in logic and mathematics there are well-established uses of "term" which a writer on the philosophy of mathematics should take into account before assigning a new significance to it. In logic a term is simply a class name; thus "man" or "men" is a term, while the phrases "every man" and "some men" are names but not terms. In mathematics a term is sometimes a name; thus a term of a polynomial is an algebraic expression. The word is however also used in connection with series, and here a term is not a name at all, but is a number or other quantity.[19] These meanings of "term" are entirely disregarded in the nomenclature put forward by Mr. Russell. He adopts none of them, but deliberately tears the word from its accepted logical and mathematical uses. Although there is already available for his purpose a suitable name "individual" (or "individual object") he wantonly disregards the law of lexicological economy, and makes "term" synonymous with "individual" and two other words besides. And this is done in a work dealing with logic and mathematics, where there is urgent need for the use of "term" in its proper senses! Mr. Russell makes no attempt to furnish a new word to be used in designating logical terms, nor does he provide us with a specific title for the mathematical terms of either series or polynomials. In all three cases he leaves us in the lurch, and in mathematics, no less than in logic, the adoption of Mr. Russell's terminology would put a great hindrance in the way of precision of thought and speech.

[19] We must warn the reader that we are here not using "quantity" in a sense of which Mr. Russell would approve. He would not call a number a quantity.

We hold, then, that philosophical inquiry has its interests best subserved by using *individual* or *individual object* where Mr. Russell prescribes the use of "term." To use "unit" in Mr. Russell's sense cannot be deemed correct, for an individual is not taken as unit in every discussion in which it enters. The fourth synonym, "entity," is as objectionable as "term," just for the very reason that its use implies every object of thought to have existence "in some sense." Or rather, as Mr. Russell himself would put it, not "existence," but "being," for he draws a distinction between the two. Speaking of any pair of "terms," A and B, he says they "need not exist, but must, like any thing which can be mentioned, have Being. The distinction of Being and existence is important, and is well illustrated by the process of counting. What can be counted must be something and must certainly *be,* though it need by no means be possessed of the further privilege of existence. Thus what we demand of the terms of our collection is merely that each should be an entity."[20] In emblazoning the word "Being" with an initial capital letter, Mr. Russell would seem to imply that this word marked something of great importance. Yet in truth the basis for ascribing Being to "anything which can be mentioned" and to "whatever can be counted" is just one insignificant fact: the fact that the name of anything which can be mentioned is subjicible, forming a sentence when some set of words beginning with "is" or "is not" is put after it; in other words, by conjoining some predicate to this name the object of thought it represents can always be said to be or not to be something. And it surely cannot be held that the mere subjicibility of a name confers, on the object this represents, a mysterious something called "Being."[21] Are

[20] P. 71.

[21] The question of existential import of propositions (which however Mr. Russell does not appear at all to have in view) brings up other considerations. It can be plausibly contended that when we enunciate an affirmative propo-

we to take purely verbal reasons for our ground, and say that the sea-serpent being inexistent, and the author of the *Principles of Mathematics* being existent, both must be put under a common category of objects possessing Being? No advantages attend this course, and there is no reason why we should adopt it. It would substantially be arbitrarily agreeing to use "has Being" in the sense of "can be mentioned." And clearly the statement that "Such and such an object of thought has Being" is utterly futile if it merely means, as it does in Mr. Russell's phraseology, that the object of thought can have a name given to it. If such modes of speech as "has being," "is an entity," etc., are to convey anything worth putting into words we must have a dichotomization of objects of thought into those that have being and those that do not—into entities and non-entities. But apart from the dichotomization marked by the names "existent objects" and "inexistent objects" there seems to be none in connection with which the words "being" and "entity" can be conveniently used. So these words are best made synonymous respectively with "existence" and "existing object." Any object of thought can, of course, be spoken of as having existence in a hypothetical universe, and in talking of that universe we can call the object an entity; but we must not on this account call such an object an entity when there is under discussion any other universe (either the actual one or a new hypothetical universe). The right to be designated as an entity is given, and given only, by existence in the universe under discussion. To take any other view would result in depriving us of a very useful

sition with a certain name as subject we thereby, in certain cases, assert the object to which this name refers to have existence. Indeed the doctrine that affirmative propositions have existential implications, but negative not, is a part of the Aristotelian logic—at least the canons of that logic will not all hold unless this be true of A, E, I and O. Evidently however reservations must be made for such propositions as "So and so is a nonentity," it being denied that the affirmative form here gives an implication of existence, or else propositions of this character must be ruled out of philosophical language entirely.

word, for we could then no longer mark out by the aid of "entity" the existent objects of a specified universe from the non-existent.

In classifying objects as entities and non-entities one has most frequently the actual universe in view. In any event however it is from a cross classification that we obtain the two classes: individual objects and collections of objects. Whether an object is to be regarded as individual or as a collection depends to some extent upon the exigencies of the moment. Thus it is often convenient to take a collection of soldiers collectively as an individual regiment or as an individual army. An object again may be a perception (under which head we would put all states of consciousness) or an object whose existence is founded on a perception or a more or less complicated set of perceptions. Under this second head come sounds (and hence spoken syllables, words and sentences), sights (and hence written letters, words and sentences), etc. Here also come all bodies and the attributes of bodies as well as various other objects which are greater removes from their elemental perceptions. Conceptions are a species of perceptions. A name may denote a conception, and moreover a name which does not do this may have a conception corresponding to it and to the object it denotes. With each type of names that do not denote conceptions there arises a question as to what sort of conceptions, if any, corresponds to names of this type, and the debate over such a problem has not seldom given rise to a discussion classical in philosophical history. For instance, one school of philosophers would say that when a man speaks of any particular triangle he has in his mind a conception corresponding to this triangle and to the individual name that denotes it, and that moreover he may have a conception of a triangle that is "neither oblique nor rectangle, neither equilateral, equicrural nor scalenon, but all and none of these at once"

corresponding to the general class name *triangle*. Another school, while admitting the existence of conceptions corresponding to proper individual names (i. e., names put forward as each denoting a single object), even when such a name is the name of a class (a collection which is here taken collectively as an individual), would deny altogether that a general name could have any conception corresponding to it; one might have, they would say, a conception of a particular triangle, or of a collection comprising many triangles, but none of triangles in general. Mr. Russell does not stop to take up any such question, but plunges at once into a classification of individual objects ("terms") which is replete with confusion. "Among terms," says Mr. Russell, "it is possible to distinguish two kinds, which I shall call respectively *things* and *concepts*. The former are the terms indicated by proper names, the latter those indicated by all other words. Here proper names are to be understood in a somewhat wider sense than is usual, and things also are to be understood as embracing all particular points and instants, and many other entities not commonly called things. Among concepts, again, two kinds at least must be distinguished, namely those indicated by adjectives and those indicated by verbs. The former will often be called predicates or class-concepts; the latter are always or almost always relations."[22] Here the name "adjective" would seem to be applied to *words* of certain species, but just two pages previous we find Mr. Russell states: "What we wish to obtain is a classification, not of words, but of ideas; I shall therefore call adjectives or predicates all notions which are capable of being such, even in a form in which grammar would call them substantives." Any comment on the confusion here exhibited between language and what language conveys would be superfluous. Such confusion is

[22] P. 44.

habitual with the author of the *Principles of Mathematics*; indeed we might say it is the corner-stone of his system of "philosophy."

Searching for more light upon what Mr. Russell means by "concept," two remarkable discoveries will be made by the patient reader: first, that concepts *occur in propositions*; second, that concepts *denote*. "A concept *denotes* when, if it occurs in a proposition, the proposition is not about the concept, but about a term connected in a certain peculiar way with the concept. If I say 'I met a man' the proposition is not about *a man*: this is a concept which does not walk the streets, but lives in the shadowy limbo of the logic-books. What I met was a thing, not a concept, an actual man."[23] The old philosophical view was that a concept was a state of consciousness. Mr. Russell's concepts, it seems, are to be found, not in the mind, but in the "shadowy limbo of the logic-books." In the logic-books themselves we can find nothing but words. Since Mr. Russell declares that "a proposition, unless it happens to be linguistic, does not itself contain words: it contains the entities indicated by words,"[24] and as he puts *word* and *concept* in antithesis, saying that we "employ words as symbols for concepts,"[25] he apparently does not regard a concept as a word or set of words. Had we been able to find any "shadowy limbo" in the neighborhood of the logic-books on our shelves we should have bravely ventured into it, and gone through those regions where that dreadful monstrosity, Symbolic Logic, "has its lair,"[26] seeking to encounter Mr. Russell's mysterious concepts, even at the peril of finding them to be Jabberwocks. But the shadowy limbo is not to be found, and filled with disap-

[23] P. 53. On this page *a man*, we are told, is a "concept," but on page 54 Mr. Russell informs us that "*A man*, we shall find, is neither a concept nor a term"! Here he not merely contradicts his statement on the previous page, but puts "concept" and "term" in an antithesis which is assuredly not in accord with his view that concepts are a species of terms.

[24] P. 47. [25] P. 53. [26] P. 66.

pointment we are compelled to abandon the hope of thus ascertaining what Mr. Russell means. At all events a concept, whatever that may be, can "occur in a proposition."

The significance Mr. Russell gives to "proposition" goes hand in hand with his peculiar use of "term." He would base a formal definition of the former word upon the motion of implication: "It may be observed that, although implication is indefinable, *proposition* can be defined. Every proposition implies itself, and whatever is not a proposition implies nothing. Hence to say 'p is a proposition' is equivalent to saying 'p implies p'; and this equivalence may be used to define propositions."[27] A less technical presentation of Mr. Russell's view is his statement that "A proposition, we may say, is anything that is true or that is false."[28] He also tells us that "propositions are commonly regarded as (1) true or false, (2) mental. Holding as I do that what is true or false is not generally mental, I require a name for the true or false as such, and this name can scarcely be other than *proposition*."[29] Most students of philosophy will be inclined to demur at the remark that propositions are commonly regarded as mental. Rather, they will say, the common practice is to regard a proposition as a written or spoken sentence, corresponding to which there may be a mental *judgment*. Here however this question is of no great importance; we are concerned, not with the practice that is prevalent, but with Mr. Russell's alone. As we have seen, Mr. Russell appears to accept "linguistic propositions" only as an unimportant species of the genus propositions, and when he speaks of propositions not mental he does not seem to have these specifically in view. Given a written or spoken sentence concerning Socrates, Mr. Russell would not ordinarily direct his attention to this sentence in considering what he

[27] P. 66 [28] P. 15. [29] P. ix.

calls the proposition. In the "proposition" which Mr. Russell would ordinarily have in view there occurs, Mr. Russell would say, Socrates himself, and not merely the name of Socrates.[30] *True proposition,* then, in Mr. Russell's sense would seem to mean *fact*. But while one can concede that John Smith may be intelligibly said to occur in the fact or state of affairs described by the sentence, "John Smith is chopping wood," when this sentence is true, it is not easy to see how John Smith (or Socrates) himself can "occur" in anything at all corresponding to a false statement about him. If Mr. Bertrand Russell, by some esoteric method, has discovered that Socrates himself, not merely the name or conception of Socrates, actually occurs in *something* corresponding to each false statement about Socrates, then it is most unfortunate that Mr. Russell's investigations have not enabled him to inform his readers precisely what this something is. And we must record our opinion that Mr. Russell's use of "proposition" is well fitted for only one purpose: the promoting a confusion between a sentence and the fact or fancy with which that sentence is conversant, just as his use of "term" is eminently adapted to create confusion between a name and what that name denotes.

Let us now return to the doctrines expounded by Mr. Russell in matters really mathematical. Part III of his work is devoted to "Quantity." The sense in which he uses this word is somewhat peculiar. He does not take it in what is probably its most suitable use, viz., its use as a generic name applicable at once to concrete numbers, to such denominate numbers as lengths, weights, areas, vol-

[30] We have already cited Mr. Russell's statement that "a proposition, unless it happens to be linguistic, does not itself contain words: it contains the entities indicated by words." On the same page he refers to "the confusion," which he says arises from "the notion that *words* occur in propositions, which in turn is due to the notion that propositions are essentially mental and are to be identified with cognitions." And on page 45 he tells us that "Socrates is a thing, because Socrates can never occur otherwise than as a term in a proposition."

umes, etc., and to the natural numbers and the other abstract quantities analogous to these (e. g., those of such values as 0, $\frac{1}{2}$, π, -1, $+\sqrt{-1}$, etc.). On the contrary he puts "number" and "quantity" in antithesis, and speaks of "applying numbers to quantities."[31] He also puts an antithesis between "quantity" and "magnitude." "An actual foot rule," he tells us, "is a quantity; its length is a magnitude.... when two quantities are equal they have the *same* magnitude."[32] Here we see again in evidence the blunder of thinking that like attributes are identical; that, for example, two bodies equal in length have the same length. Indeed it is upon this blunder that Mr. Russell founds his criterion for distinguishing between quantities and magnitudes. "A quantity is anything which is capable of quantitative equality to something else." "A magnitude.... is to be defined as anything which is greater or less than something else." And there is, Mr. Russell holds, "reason to think.... that what can be greater or less than some term can never be equal to any term whatever, and *vice versa.*"

With number attributes, as we have seen, Mr. Russell does not even put the question whether what are colloquially called the same attribute common to different groups ought not more properly to be regarded as like but different attributes. With "magnitudes" however he does give some attention to what he calls "the relative theory" which denies that there is any such thing as "a magnitude shared by equal quantities."[33] His "refutation" of this theory is worthy of note. He lays down eight indemonstrable axioms which he says the theory obliges us to assume, one of these

[31] P. 157. He tells us, page 159, that "In fixing the meaning of such a term as *quantity* or *magnitude,* one is faced with the difficulty that however one may define the word, one must appear to depart from usage." This is one of the many instances in which he slips away from the meaning he nominally gives "term," and uses the word in the customary logical sense.

[32] P. 159. On page 165 he tells us that "all magnitudes are simple concepts."

[33] P. 162.

axioms affirming that, whenever A is a quantity, "A being given, there is always a B [likewise a quantity] which may be identical with A, such that A = B," or, as he also puts it, "If A be a quantity, then A = A." He tells us that "These axioms, it will be observed, lead to the conclusion that, in any proposition asserting equality, excess, or defect, an equal quantity may be substituted anywhere without affecting the truth or falsity of the proposition. Further the proposition A = A is an essential part of the theory. Now the first of these facts strongly suggests that what is relevant in quantitative propositions is not the actual quantity, but some property which it shares with other equal quantities. And this suggestion is almost demonstrated by the second fact A = A. For it may be laid down that the only unanalyzable symmetrical and transitive relation which a term can have to itself is identity, if this indeed be a relation."[34] And Mr. Russell goes on to show—to his own satisfaction, though to us his argument is by no means cogent—that the admission of equality as a symmetrical transitive relation which an individual object ("term") can have to itself leads to a theory of magnitude which is not the "relative theory" at all, but the "absolute theory" that he himself upholds. We shall not here consider whether the first "fact" is really a fact—whether so broad a statement as to the substitution of [symbols of] equals can properly be made in an algebra developed on a philosophical basis. For in any event we are quite unable to see that this "first fact" can give the suggestion which Mr. Russell accredits to it. And as for the "second fact," on which Mr. Russell relies for the definite refutation of the "relative theory": the necessity in that theory for a quantity to be equal to itself, this may really be essential to the theory elaborated by Mr. Russell's fancy as a straw man for him to overthrow. But it cer-

[34] P. 163.

tainly is not the case that an algebra sanctioned by a true philosophy will ever affirm A to equal A. This affirmation is seen not to express a fact at all on making the proper distinction between equality and identity, and recognizing that the mathematical character $=$ does not always concern true equality, but has sometimes reference to identity. It is absurd to say that because mathematicians write $A = A$ and read this "A equals A," a quantity must necessarily be equal to itself. A philosopher ought not to take the customs of mathematical symbolism as foundation for his doctrines.

Another statement of Mr. Russell's bearing on the import of $=$ is of interest enough to cite here. "Among magnitudes, equality...has an absolutely rigid and unique meaning: it applies only to quantities, and means that they have the *same* magnitude.... Among numbers....there is no such thing as equality. There is identity, and there is the relation which is usually expressed by the sign of equality, as in the equation $2 \times 3 = 6$. This relation had puzzled those who endeavored to philosophize about arithmetic, until it was explained by Professor Peano. When one term of the equation is a single number, while the other is an expression composed of two or more numbers, the equation expresses the fact that the class defined by the expression contains only one term, which is the single number on the other side of the equation."[35] As a matter of fact, the explanation which Mr. Russell ascribes to Professor Peano is wholly erroneous. The numeral 6 does not denote a single number: it denotes many numbers; all

[35] P. 341. With his habitual confusion of thought and speech, Mr. Russell here leaves us in doubt as to whether the equation is a sentence written symbolically with numeral expressions as the two members, or whether it is something that the sentence merely serves to express, its "terms" being what the members of the sentence represent. Indeed his words might be fairly construed to imply that the equation is made up of a number (be it noted that he does not say numeral or symbol of a number) on one side and a mere expression on the other, the expression in some strange way being "composed of two or more numbers"!

those comprised in the value class of the sixes. Likewise the expression 2×3 denotes a multitude of numbers, all of which are sixes, but not all sixes are denoted by this expression. For not every six is the result of an operation of multiplying a three by a two, and only the numbers resulting from such operations are entitled to be designated by the expression 2×3. And $2 \times 3 = 6$ means: Every product of a two into a three is equal to or identical with every six (i. e., is equal to every six save that with which it is identical). So here $=$ has reference to both equality and identity.

From quantities[36] we may proceed to the consideration of variables. The conception of a variable is a most important one in mathematics, and can be traced back to the attempts made by Archimedes and other Greek mathematicians at the rectification and the quadrature of circles. In this work it was essential to conceive of a variable and the limit which the latter approached, though it was not until many centuries later that "variable" or any equivalent name was heard of in mathematics. The use of the word does not date back to much before the time of Newton, and the earlier definitions all ascribed to a variable the character of a quantity, quantities being classified as constants and variables. Definitions of this type are by no means obsolete. Thus in a work on differential and integral calculus, published not very many years ago under the auspices of Professor Peano, we are told that "In the questions considered there may appear quantities to which determined and fixed values are supposed to be attributed, and these are called constants, and other quantities supposed to be able to assume diverse values, and these are called variables."[37] An examination however of the vari-

[36] The word "quantity" we shall hereafter use, not in Mr. Russell's sense, but in what we esteem to be the proper one.

[37] *Calcolo Differenziale e principii di Calcolo Integrale* by A. Genocchi, "Publicato con aggiunto dal Dr. Giuseppe Peano."

ables that mathematics takes within its scope shows that such a definition is quite unsuitable. The question as to whether a variable can properly be termed a quantity is closely related to the problem of sameness and similarity of attributes, possessed by different groups or different individuals, which arises in connection with the natural numbers and other quantitative attributes (e. g., lengths, weights, volumes, areas, etc.). We decided in the foregoing pages that two groups alike in number ought not to be said to have the *same* number, but to have different though *equal* numbers, and that two bodies alike in length ought not to be said to have the *same* length, but to have different though *equal* lengths. And so with weights, volumes, areas, etc.

Suppose now that there is only a single body in view, and that a change takes place in reference to one or more attributes. Suppose, for instance, that a bar of metal is heated, and its length changes from 1000 centimeters to 1020 centimeters, while its color changes from black to red. Are the facts well expressed by saying the bar possesses the "same" length and the "same" color as before? If the old definition of variable is to be taken as criterion, we must answer in the affirmative as regards the length at least. For the length of the bar during the change is what would be called a *variable* in mathematics, and the definition in question tells us a variable is a quantity—that is, *one* quantity. And should we take a like ground in reference to color attributes we must say that the manifold colors which appear throughout the change are a color, that is, *one* color. If however we refuse to be bound by an ancient definition, and prefer the only course sanctioned by a sound philosophy—that of so using language as to mark the distinctions to which our senses testify, then we must recognize that such a use of "same," implying identity where there is diversity, is even more repugnant than

its application in the sense of equal. And we see that when an object initially possesses a certain attribute and undergoes a change with respect to this attribute, we must then regard as a distinct and separate attribute each stage of the process of change.[38] Such a set of attributes may together constitute a variable, and under this description come most of the variables of physical science. We may have a body with variable velocity—a type of variable which gave frequent opportunity for the exercise of Newton's genius—a body of gas with variable temperature, under variable pressure and with variable volume. We cannot say that any of these variables are quantities unless we wish our language to be an impediment, instead of an aid, to exact thinking.

Even stronger do we find our case when we turn to geometry, and consider the type of variable that first appears in history. Archimedes, in striving to rectify a circle, inscribed in it first an equilateral triangle, then a regular hexagon, then a regular dodecahedron, etc. The perimeters of the inscribed polygons here constituted a variable whose limit was the circumference of the circle, while the areas of the polygons constituted a variable whose limit was the area of the circle. With what show of reason can we say that the perimeter of a triangle is one and the same quantity—one and the same perimeter—as the longer perimeter of a hexagon? How are we justified in saying that the area of a triangle and the area of a much larger ninety-six-sided figure are the same quantity? And yet we must make these assertions if we acknowledge that a

[38] The stages recognized by our naked organs of sense are comparatively few. The various instruments at our disposal for assisting the senses enable us greatly to increase the number of stages that can be deliminated, but this number always remains finite. To assume that a succession of suitable instruments would, if they were at hand, enable us to keep up a continual increase *ad infinitum* in the number of stages detected, and hence to ascribe innumerable stages to the change where we really do not observe so many (which is in effect what is usually done in scientific work) is to adopt a hypothesis, legitimate enough as such, but not to be taken as anything more than a hypothesis.

variable is a quantity. Nor is it only with circles that such difficulties arise. The quadrature (or rectification) of any curvilinear figure by the methods of the modern integral calculus presents similar difficulties. We have a set of rectilinear figures, no two alike, the areas of which (or the lengths of certain lines on which) constitute a variable the limit of which is sought. Surely these considerations are overwhelming against the doctrine which would ascribe to a variable the character of a quantity, and there is no recourse save to abandon that doctrine completely.

A variable, then, is not a quantity, but is constituted by a set of quantities, and the quantities in the set may be natural numbers or other quantities of the same sort, or be concrete quantities, or be such denominate quantities as lengths, areas, etc., It remains for us to ascertain the essential characteristics of such a set, for sets of quantities which do not constitute variables also come under consideration in mathematics. Now the characteristic investigations in which variables appear in mathematics are undoubtedly those where limits are concerned. And it is to be noted that a limit (which is a quantity that may or may not itself belong to the variable) pertains, not to any one quantity in the variable taken separately, but either to the whole set of quantities or to a part of that set so extensive as to comprise innumerable quantities. When it is said that the variable x approaches the limit l, it is not meant that any quantity of x approaches l; such a statement would be utterly meaningless. If, for instance, we have a variable composed of unique representatives of the values 1, 1½, 1¾, 1⅞, etc. arranged in the order here given, then this variable approaches as limit a quantity of value two (a limit which is in this case not in the variable) but it would be absurd to speak of the 1 or of the 1½, or of any other quantity of the variable as approaching a limit.

Again a variable composed of unique representatives of all real abstract values from 1 to 3 arranged in order of value first approaches a limit of value two (which is here in the variable), then attains it, and finally recedes from it, but we cannot say that any separate quantity of the variable approaches or recedes from this or any other limit. A variable may approach a limit throughout its whole extent or throughout only a portion of its range, but in no event can a quantity of the variable be said to approach a limit. And we may lay down that the purpose of taking a set of quantities together as constituting a variable is to investigate matters that concern the mutual relations between quantities of the set, this being the case whether a limit is or is not concerned. Here then is found the essential characteristic of the set of quantities which constitutes a variable: the purpose for which it is formed. There is a sharp contrast in this respect between the sets of quantities comprised in a variable and other sets of quantities which appear in mathematics. Take, for example, the value classes. When we form a value class, and give it a class name, as when we group together all the twos, and provide the class name "two," our purpose is to investigate questions and state propositions concerning every two taken separately or concerning each of some of the twos. When we say $2 + 3 = 5$, we assert that every two plus any three is equal to or identical with every five, an assertion wholly unlike that made in "The variable x approaches the limit l."

On first thought it might appear appropriate to specify, as another distinguishing feature of variables, the taking of diverse values. It seems only natural to say that there must be variation of value in a variable. But to take etymology as decisive in laying down a definition is the method of the pseudo-scientiest, not that of the really scientific investigator. On inquiry into the matter we indeed find that variation is not always at hand. Mathematicians have

even coined a specific name, *tratto d'invariabilita* in Italian, *Invariabilitätszug* in German, to designate a portion of a variable (here the dependent variable of a functional relation) in which there is conformity to a single value, and not variation. And moreover there are variables with which variation of value is everywhere lacking. For in analytical geometry the mathematician finds it convenient to regard the abscissas of a line parallel to the y axis (or the ordinates of a line parallel to the x axis) as constituting a variable, though obviously in such a variable all the quantities are of a single value.

While in many mathematical works the ancient definition of variable still lingers on, a new type of definition is beginning to appear in some of the more modern treatises. The innovation is however unobtrusively set down without any word to show that a conception of a variable is being put forth which is different from that embodied in the old definitions. These new definitions describe a variable as a symbol. Thus in that most authoritative of mathematical works, the *Encyklopädie der mathematischen Wissenschaften,* Professor Pringsheim, in his article on the *Grundlagen der allgemeinen Funktionenlehre*, lays down that "By a real variable is to be understood a *symbol,* usually one of the letters of the alphabet, to which is assigned successive different number-values (for example, all possible between two fixed number-values, all rational, all integral)."[39] The only way in which this definition has reference to real variables as opposed to variables in general, is by the use of "number-values" (i. e., real values[40]) instead of simply "values." So in Pringsheim's view a variable would seem to be essentially a symbol to which there are assigned in succession various different

[39] Vol. II, Part I, p. 8.

[40] "Numerical value" would of course not be a correct translation of *Zahlenwerte,* though this is sometimes so translated when used in just the sense taken for it by Pringsheim.

values. To those who are satisfied to accept manipulation of symbols as the Ultima Thule of mathematics, defining a variable as a symbol will doubtless be satisfactory. Such a course produces quite a different impression however upon those who would no more take as the subject matter of mathematics the words and symbols it makes use of than they would rest satisfied with a science of botany which studied such "words" as *coniferae and cruciferae,* and did not investigate the objects these words stand for. Variables in mathematics are not symbols any more than numbers or other quantities are symbols. Variables and quantities are both things represented by symbols. Pringsheim's definition and others of the same type have really reference, not to variables, but to the names or symbols of variables. So construed they seem to mean that when and only when a symbol takes successively a number of different values (that is to say, denotes in succession quantities of these respective values) then that symbol is the symbol of a variable. How very remote this view is from the actual facts in the case is sufficiently obvious from the investigations into the constitution of variables that have been made in the foregoing paragraphs.[41] The most characteristic use of the name or symbol of a variable is, we have seen, not in propositions about the quantities of the variable taken individually, and in the characteristic use (e. g., when we say x approaches a limit) the symbol *does not take any value at all.* It is true that mathematics also sanctions using the symbol of a variable in quite another way—in the equations dealing with functional relations—but this latter use is by no means distinctive of the sym-

[41] One or two mathematicians cursorily define a variable as an *aggregate* of quantities, but go no further into the question of the characteristics of a variable, and say nothing of the distinctive way in which the name of a variable is used. Such a definition is not, on its face, so untenable as the quantity definitions or the symbol definitions, but we cannot regard it as entirely satisfactory. For a discussion of this matter however we must refer the reader to our forthcoming work: *Fundamental Conceptions of Modern Mathematics.*

bols of variables. And before we consider it we shall proceed to see what doctrine Mr. Russell holds as to the nature of variables.

In Mr. Russell's system variables are of prime importance. The reader is introduced to them at the very outset of the *Principles of Mathematics.* "Mathematical propositions," Mr. Russell says, "are not only characterized by the fact that they assert implications, but also by the fact that they contain *variables.*"[42] In his use of the term "variable" Mr. Russell is not content to abide by the ordinary mathematical customs, but seeks to give a much broader field to its application. "It is customary in mathematics to regard our variables as restricted to certain classes: in arithmetic, for instance, they are supposed to stand for numbers. But this only means that *if* they stand for numbers, they satisfy some formula, i. e., the hypothesis that they are numbers implies the formula. This, then, is what is really asserted, and in this proposition it is no longer necessary that our variables should be numbers: the implication holds equally when they are not so. Thus, for example, the proposition 'x and y are numbers implies $(x+y)^2 = x^2 + 2xy + y^2$' holds equally if for x and y we substitute Socrates and Plato. [It is necessary to suppose arithmetical addition and multiplication defined (as may easily be done) so that the above formula remains significant when x and y are not numbers]: both hypothesis and consequence in this case, will be false, but the implication will still be true. Thus in every proposition of pure mathematics, when fully stated, the variables have an absolutely unrestricted field: any conceivable entity may be substituted for any one of the variables without impairing the truth of our proposition."[43]

[42] P. 5.

[43] P. 6. It should be remembered that in Mr. Russell's view "propositions" are not sentences, and that in a proposition "occur," not names, but the things and concepts designated by names. Hence to substitute the entity Socrates

Even in ordinary mathematics Mr. Russell gives variables a rôle much more important than that usually assigned to them: "The variable is, from the formal standpoint, *the* characteristic notion of mathematics. Moreover it is *the* method of stating general theorems.... That the variable characterizes mathematics will be generally admitted, though it is not generally perceived to be present in elementary arithmetic. Elementary arithmetic, as taught to children, is characterized by the fact that the *numbers* occurring in it are constants; the answer to any schoolboy's sum is obtainable without propositions concerning *any* number. But the fact that this is the case can only be proved by the help of propositions about any number, and thus we are led from schoolboy's arithmetic to the arithmetic which uses letters for numbers and proves general theorems.... Now the difference consists simply in this, that our numbers have now become variables instead of being constants. We now prove theorems concerning n, not concerning 3 or 4 or any other particular number... Originally, no doubt, the variable was conceived dynamically, as something which changes with the lapse of time, or, as is said, as something which successively assumed all values of a certain class. This view cannot be too soon dismissed. If a theorem is proved concerning n, it must not be supposed that n is a kind of arithmetical Proteus, which is 1 on Sundays and 2 on Mondays, and so on. Nor must it be supposed that n simultaneously assumes all values. If n stands for any integer, we cannot say that n is 1, nor yet that it is 2, nor yet that it is any other particular number. In fact, n just denotes *any* number, and this is sometimes quite distinct from each and all of the numbers. It is not true that 1 is any number, though it is true that whatever holds of any number holds of 1. The variable,

for x in a proposition is not, it would seem, to substitute the *name Socrates* for the symbol x, but to actually take Socrates out of his grave, and put him, in some incomprehensible way, in the place of the object represented by x!

in short, requires the indefinable notion of *any*."[44] This "indefinable notion of *any*" is put by Mr. Russell in sharp contrast to the notions of "all" and of "every." "*All a*'s denotes a numerical conjunction.... The concept *all a*'s is a perfectly definite single concept, which denotes the terms of *a* taken altogether.... *Every a*, on the contrary, though it still denotes all the *a*'s, denotes them in a different way, i. e., severally instead of collectively. *Any a* denotes only one *a*, but it is wholly irrelevant which it denotes, and what is said will be equally true whichever it may be. Moreover *any a* denotes a variable *a*, that is, whatever particular *a* we may fasten upon, it is quite certain that *any a* does not denote that one; and yet of that one any proposition is true which is true of any *a*."[45]

Mr. Russell further tells us: "We may distinguish what may be called the true or formal variable from the restricted variable. *Any term* is a concept denoting the true variable; if *u* be a class not containing all terms, *any u* denotes a restricted variable. The terms included in the object denoted by the defining concept of a variable are called the *values* of the variable: thus every value of a variable is constant."[46]

Finally we are told what a variable is: "Thus *x* the variable is what is denoted by *any term*."[47] "Thus *x* is in some sense the object denoted by *any term*, yet this can hardly be strictly maintained, for different variables may occur in a proposition, yet the object denoted by *any term*, one would suppose, is unique. This however elicits a new point in the theory of denoting, namely that *any term* does not denote, properly speaking, an assemblage of terms, but denotes one term, only not one particular definite term. Thus *any term* denotes different terms in different places."[48] "The notion of the variable.... is exceedingly complicated. The *x* is not simply *any* term, but any term with a certain

[44] P. 90. [45] P. 58. [46] P. 91. [47] P. 6. [48] P. 94.

individuality; for if not, any two variables would be indistinguishable."[49]

Mr. Russell's extension of the word "variable" to cases where what are in question are not quantities is quite in line with his other innovations in terminology. At all of these we are inclined to demur. We shall not however dwell upon such matters here, but will restrict ourselves to inquiring whether Mr. Russell's doctrine, in its application to cases where quantities do happen to be concerned, conforms to the true theory of the variables met with in actual mathematical work. Now it is quite evident, on examining Mr. Russell's remarks, that his idea of a variable has not arisen from a systematic consideration of the variables of mathematics. Mr. Russell's conception of a variable is essentially an etymological one, and what he takes to be variables are not variables at all. We have shown that, with a variable, variation of a value is neither sufficient nor necessary. Mr. Russell however thinks that taking different values is the very essence of a variable. In this view, "If x and y are numbers, $(x+y)^2 = x^2 + 2xy + y^2$" is a typical case of a proposition involving the symbols of variables. And he holds that whenever we formulate a proposition concerning n instead of merely concerning 1 or 2 or 3 or some other particular number then we are dealing with variables. The facts are, to begin with, that a proposition about what Mr. Russell calls a particular number, for example 3 in the equation $3 + 2 = 5$, is about a whole species of numbers—the value class of the threes, or rather it is about every number in this species taken individually. And when we go to a general theorem involving such a symbol as n we merely ascend to a genus comprising several (usually innumerable) value classes. When n is used in what Mr. Russell describes as a proposition about any number, it is a class name denoting every number of every

[49] P. 107.

value class whatsoever, and the propositions to which Mr. Russell refers as "about *any* number" may with perfect propriety be said to be about each and every number. Take, for instance, the equation $(n + 1)(n - 1) = n^2 - 1$. This affirms: *With every number whatsoever the sum of this number plus any one, multiplied into the difference of the same number minus any one, is equal to or identical with the remainder obtained by subtracting any one from the square of the number in question.* It is true that we can read the equation as a proposition in "any number" if we prefer to do so. We can render it: *The sum of any number plus any one, multiplied into the difference of the same number minus any one, is equal to or identical with the remainder obtained by subtracting any one from the square of the number in question.* But there is no diversity of import between this and the rendition first given. The "any" version and the "every" version have precisely the same meaning. The difference is not a philosophical one, but resides merely in the syntactical construction of the one sentence requiring "any" (or "each," which could just as well be used) while the syntax of the other permits the use of "every." In taking the curious view that a general theorem of arithmetic involving such a symbol as n is characterised by being "about *any* number," and in implying, as he does by the opposition in which he puts "any" and "every," that it is not about *every* number, Mr. Russell is propounding a doctrine which is utterly untenable. Between "any" and the n (or other symbol) of a general theorem there is not really the connection imagined by Mr. Russell, and the vague subtleties which fill several pages devoted to "any" in the *Principles of Mathematics* are beside the point. What connection there is lies between the symbol and the whole set of universal syncategorems,[50]

[50] We use "syncategorem" as the title of those words and phrases which are conjoined to class names to indicate the logical quantity. Sometimes "syn-

every, each, any, etc., and does not pertain to any one of the set to the exclusion of the rest. As regards the class to which such a symbol refers the theorem is universal—is a proposition in which the class name of that class is distributed—and if it is to be put into words some universal syncategorem must be conjoined to the class name to indicate the distribution. In the purely symbolic mode of expression sanctioned by mathematics there are however no characters representing syncategorems; these are always taken for granted, mathematics dealing so largely with generalities that whenever a formula occurs the tacit assumption is made that this must be universal with respect to every symbol involved. These symbols are class names to which syncategorems must be conjoined when we pass from formulas to equipollent sentences in which we retain the symbols but none of the signs or other characters of the originals. And thus the *n*, in a proposition about every number ("about *any* number," as Mr. Russell would put it), is a synonym of "number"; it is a class name for numbers of all species. Like other class names, it by itself denotes every member of its class, while when conjoined to a syncategorem every member is or is not denoted, according to whether the syncategorem is or is not universal. In the general theorems of mathematics, since a universal syncategorem is understood though not expressed, every member of the class, that is, every quantity (or every quantity of the type in question, e. g., every number) is denoted, and hence the symbol does simultaneously assume all values (or all values of the type), Mr. Russell to the contrary notwithstanding.

The confusion of such a symbol as Mr. Russell's *n*—a general symbol for quantities of many values—with the symbol of a variable is not peculiar to Mr. Russell, and

categorematic word or phrase" is used in a broader sense in which it applies also to certain words that appear as *parts* of class names, e. g., of, in, at.

seems to be largely due to a one-sided view of the symbol of a variable. There are in mathematics important propositions in which symbols of variables are used in a way which enables them to be construed as here denoting the quantities of variables, just as the class name belonging to a class of quantities denotes the quantities of that class. These propositions are the equations concerned with functional relations between two or more variables. Thus y and x being two variables, there might be a functional relation between these that would in mathematical symbolism be indicated by $y = x^2$, the significance of this equation being that a functional relation[51] subsists under which every quantity in the variable y which has a corresponding quantity in the variable x is equal to or identical with the square of the corresponding quantity in x. And if we choose, we may use the shorter phrase "every y"[52] instead of "every quantity in y," and say "every x" instead of "every quantity in x." With this mode of reading, if one considers only such equations and neglects the more characteristic use of the name or symbol of a variable, it is not surprising that he should confound the names of variables with ordinary class names. There is really an analogy between such a proposition as "Every y having an x corresponding to it is equal to or identical with the square of

[51] We hold that a functional relation between two variables is established whenever there is, first, correspondence between quantities of the two variables and, second, likeness in order of corresponding quantities. It is not essential that *every* quantity in either variable should have a corresponding quantity in the other. For a discussion of this matter we refer to the work already mentioned.

[52] Here y does take values. But the way in which it does so is not by having successive different values assigned to it as Pringsheim wishes us to believe. It does not assume the values represented in the variable *successively*; it takes all of them *simultaneously*. There may be consecution connected with a variable, but it is not a consecution of the times in which a symbol takes different values; it is a consecution in the order of arrangement of the quantities constituting the variable. For the quantities of the variable have commonly an arrangement in order—an arrangement which with some variables is immutable but with others is not. An arrangement under which the quantities are consecutive may, as we have said, occur, but other arrangements are common, and some variables are not even capable of having their quantities arranged in a sequence. So it is certainly not a happy thought to bring succession into the definition of a variable.

the corresponding x," and a proposition in which occurs the class name of a class of quantities formed solely for the study of properties which the quantities of that class possess as individuals. Be it noted however that the closest analogy is not found with those general propositions of mathematics which Mr. Russell takes as typical propositions involving the symbols of variables. Thus consider $a + b = b + a$ which, when put forward as enunciating the commutative law of addition, is a fair example of these general propositions. This should be read: "Every a when any b of the same sort[53] is added to it gives a sum equal to or identical with the sum of the same b as before plus the same a as before." Here we have what might be called an equation of sameness; in reading it we are obliged to specify that the second a taken is the same as the first a, and that the second b is also the same as the first b. Otherwise the proposition would not be true; for a and b are both general class symbols each of which takes all values and denotes every quantity that enters mathematics; and hence if we did not put the specification as to sameness we should be asserting that every sum of two quantities is equal to or identical with every sum of two quantities, whether the quantities entering the second sum were the same as those entering the first or not. Now the equations concerning functional relations do not always carry a stipulation as to sameness; this may be at hand, as in $y = x^3 - x^2$ where the same x must be squared in forming the subtrahend as was cubed to form the minuend, or it may not be needed at all as in $y = x^2$. On the other hand these equations always carry a stipulation totally lacking with $a + b = b + a$: namely a stipulation as to correspondence. For the equality and identity alternative in equations concerning a functional relation

[53] It is necessary to specify that the b taken must be of the same sort as the a. For if it is of a different sort the addition cannot be performed. Thus we cannot add a length to a weight, or add either to a natural number.

between a y-variable and an x-variable holds only of a y and of its *corresponding* x. Stipulations of correspondence, though not at hand in such equations as $a + b = b + a$, can however be found where ordinary class symbols are alone involved. The special equations that arise in connection with certain problems bear such stipulations. Take for example the system of equations $x + y - 10 = 0$, $2xy - 9x - 11y + 54 = 0$, where x and y are symbols of unknown quantities—*quesitive symbols* as we might call them. These equations are satisfied by $x = 7$, $y = 3$, and by $x = 4$, $y = 6$. Here x denotes quantities of two different values, and so does y. We cannot read $x + y = 10$ as "Every sum of an x plus a y is equal to or identical with every ten"; this is not true, for $7 + 6$ is not 10, nor is $4 + 3$. We must say "Every sum of an x plus the corresponding y," and must likewise read the other equation of the system as an equation of correspondence. In quite another way, correspondence is in evidence with the general equations each of which covers a host of special cases. Such an equation is the general equation of the second degree with one quesitive symbol: $ax^2 + bx + c = 0$, which on solution gives $x = (-b \pm \sqrt{b^2 - 4ac})/2a$. This solution, as well as the original equation, is understood to carry stipulations as to correspondence; the equations do not have reference to each set of values that can be formed from the values taken separately by the four literal symbols—if they did the first of the equations would assert such absurdities as $3 = 0$; for x, a, b and c all take the value 1. The equations are understood to hold only for each set of corresponding values; in other words the equality or identity alternative is asserted only of an x, an a, a b and a c which correspond. Mathematicians who deal with the theory of equations not infrequently use the name "variable" in connection with the x of such equations as these, but decline to speak of variables where a, b and c are concerned, thus

taking the ground that the symbol of a variable must be quesitive. They ought however to know that in what is usually regarded as the natural habitat of variables—the differential and integral calculus—it is by no means a requisite that the quantities of a variable be unknown. If concerned in a correspondence or taking diverse values or both together be accepted as criterion, then clearly *a, b* and *c* in the general equation as well as *x* ought to be designated as symbols of variables, and so likewise ought *x* and *y* in the system of special equations. But the real criterion has nothing to do with variation of value, and nothing to do with correspondence (which does not come into consideration in the constitution of a variable, but only appears on the stage when a functional relation is established between several variables); and none of these symbols are symbols of variables. A set of quantities constituting a variable is characterized, not by the values of its components or by any correspondence into which they enter, but by the purpose for which the set was formed, or, if we wish to be perfectly precise, the purpose which it serves. Take *x* in the general equation of the second degree: the purpose in view in grouping together the quantities called *x*'s is not to investigate their mutual relations, but merely to facilitate finding the value of the *x* of each instance that may come to hand as a special case of this general equation, and thus to avoid the trouble of formulating and solving a new equation for each separate case. The *x*'s (and the *a*'s and the *b*'s and the *c*'s) may be made to constitute a variable by carrying out an investigation which does bear on their mutual relations, but when initially grouped together they assuredly do not pertain to a variable at all.

In considering Mr. Russell's doctrine that the ordinary class symbols of mathematics are names of variables, we have. let the reader remember, made due allowance for

the fact that he did not well choose the examples he gave of the former, but failed to put forward those most analogous to the latter as used in equations dealing with functional relations. We have gone out of our way to bring to light the cases where the greatest analogy is to be found, thus giving the most favorable aspect to Mr. Russell's contention. And the result of our investigations is assuredly not to sanction the identification of symbols of variables and ordinary class symbols used only to denote the separate members of classes of quantities. Between the symbols of these two types there is, we see, a difference of character which makes Mr. Russell's doctrine utterly untenable. This doctrine, ignoring as it does the really characteristic use of names of variables, must be deemed quite unworthy of a philosopher. What extenuation can be urged in Mr. Russell's behalf? The only one we can conceive as being put forward is the plea that Mr. Russell had a perfect right to use "variable" in a sense peculiar to himself, and to give it, in the discipline he miscalls "mathematics," an application quite different from that which the word has in the mathematics of the mathematicians. Now we willingly concede that deviations from the established meanings of important scientific terms, frequently as Mr. Russell indulges in them, ought not to be looked upon with intolerance. We would not reproach him merely for this: such offenses against current usage are in themselves only venial sins for a philosopher. But, though they are of small moment alone, it is quite a different matter when they have as concomitant a confusion of thought which pervades a whole philosophy and seem to play no small part in bringing about this confusion. And that is precisely what we find with the innovations fathered by Mr. Russell, who, moreover, while admitting that in many cases he apparently departs widely from common usage, does not always seem aware of the extent of his

transgressions.[54] Notably with the very word "variable" Mr. Russell is to all appearances serenely unconscious of his flagrant contravention of the terminology sanctioned by scientific use. His application of the name, so far as he uses it in connection with the symbols occurring in ordinary mathematical formulas, he takes as a matter of course, though in fact nothing is more unwarranted than his procedure here. After his initial error we can hardly be surprised at his extension of this application, under which he designates as names of variables the phrases obtained by conjoining "any" to even those general class names that do not pertain to what is ordinarily called mathematics. The extension is a natural result of Mr. Russell's assigning to mathematics the broad field of symbolic logic, and indeed, the misconception of variable is taken as basis for the formal definition of "Pure Mathematics" promulgated in the *Principles*. "Pure Mathematics," Mr. Russell informs us, "is the class of all propositions of the form 'p implies q,' where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants."[55] Elsewhere he tells us, "The variable is, from the formal standpoint, *the* characteristic notion of Mathematics."[56] And he actually says that his definition of Pure Mathematics "professes to be, not an arbitrary decision

[54] Of all the words in a language "not" is one of the few whose meanings are firmly established by a usage uniformly alike in scientific and in colloquial discourse. Will it be believed that Mr. Russell is found making a use of this word utterly opposed to the meaning which has heretofore been universally assigned to it? Speaking of relations (p. 25) he notes that "with some relations, such as identity, diversity, equality, inequality, the converse is the same as the original relation: such relations are called *symmetrical*." And he then proceeds to say: "When the converse is incompatible with the original relation, as in such cases as greater and less, I call the relation *asymmetrical*; in intermediate cases, *not-symmetrical*." Needless to say, in the sense in which "not" had been universally used up to the time Mr. Russell began to write, the asymmetrical and the "not-symmetrical" relations would alike be designated as *not symmetrical*. Yet there is nothing in his remarks to show that Mr. Russell was conscious of how greatly "not" is distorted from its accepted meaning in the nomenclature he devised.

[55] P. 3.

[56] P. 90.

to use a common word in an uncommon significance, but rather a precise analysis of the ideas which, more or less unconsciously, are implied in the ordinary employment of the term."[57] The use of the word "variable" in Mr. Russell's definition may perhaps so impose upon some of his unthinking readers as to make them swallow this contention, but if that word be elided, and a less misleading synonym be put in its place, then Mr. Russell's analysis of the ideas implied in the ordinary employment of the term "Pure Mathematics" will even to the most casual reader appear anything but precise. For what Mr. Russell really means by a variable here is either a *general class name, distributed by the conjunction of a universal categorem or otherwise*, or else the object represented by the distributed class name—the hazy nature of his doctrines makes it difficult to tell which. And it is perfectly ludicrous to say that Pure Mathematics, in the ordinary acceptance of this term, is characterized by such distributed class names or by what they stand for.

We have seen then that in what are perhaps the two questions most fundamental in mathematical philosophy—the doctrine of numbers and the doctrine of variables—Mr. Russell's failure is complete and utter. His delineation of numbers is inadequate, and the definition he puts forward is inadmissible. What he regards as variables are not the variables of mathematics, nor has he penetrated into the nature of what he wrongly takes to be variables. He is to all appearances unaware that "any individual" (which in his view denotes the true variable) and "any *u*" (which he regards as denoting a restricted variable) are nothing more than distributed general class names. He assumes that "any" has always an office radically different from that of "every," and puts forth many would-be subtleties in a laborious attempt to show just how the two syn-

[57] P. 3.

categorems differ.[58] Starting from an erroneous assumption, he succeeds in bringing obscurity over a matter that is on its face perfectly clear, and is after all brought to admit that for "the variable," whose nature is, in Mr. Russell's opinion, inextricably bound up with the question of "any," a satisfactory theory is yet to be found.[59] Rightly or wrongly, after going over Mr. Russell's theories, one feels inclined to ascribe the errors he commits very largely to the curious metaphysics which here and there makes itself manifest. An inkling of Mr. Russell's metaphysics may be gained from what he propounds concerning propositions and concerning objects and terms, things and concepts. In dealing with the latter he puts forward a classification—substantially a doctrine of categories—which appears to be quite original and is likewise preposterous. Had Mr. Russell deferred his work in the field of mathematics until he had elaborated as foundation a sane and coherent doctrine on these points and others of pure philosophy, he might not have so rapidly written a ponderous volume, but assuredly he would have been more likely to have produced a work of real benefit to science. As it is, one can only regret that the author of the *Principles of Mathematics* should have expended so much labor in erecting an imposing edifice upon foundations that are thoroughly unsound.

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[58] In fact, as we have said before, opposition between "any" and "every" is often quite lacking, and, an "any" sentence can in such a case be changed into an "every" sentence with but slight modification. However opposition does sometimes occur, but not in the way Mr. Russell would have us think. Opposition occurs with the potential mood; thus, "Any horse may win the race" is truly opposed to "Every horse may win the race." When we pass to the affirmative indicative and compare, for example, "Every horse is a mare" and "Any horse is a mare," we are no longer able to find opposition; the latter sentence is not of a usual form, but if we grant it any meaning we will naturally accept it as equipollent to the former. With the negative indicative there is however opposition, "Not every horse is a mare" and "Not any horse is a mare" being opposed,. (We need hardly say that we do not use "opposed" in the sense of "incompatible.")

[59] P. 5.